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From Integers to Fractions: The Role of Analogy in Developing a Coherent Understanding of Proportional Magnitude

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Children display an early sensitivity to implicit proportions (e.g., 1 of 5 apples vs. 3 of 4 apples), but have considerable difficulty in learning the explicit, symbolic proportions denoted by fractions (e.g., "1/5" vs. "3/4"). Theoretically, reducing the gap between representations of implicit versus explicit proportions would improve understanding of fractions, but little is known about how the representations develop and interact with one another. To address this, we asked 177 third, fourth, and fifth graders (M = 9.85 years, 87 girls, 69% White, 19% low income) to estimate the position of proportionally equivalent integers and fractions on number lines (e.g., 3 on a 0–8 number line vs. 3/8 on a 0–1 number line, Study 1). With increasing age, children's estimates of implicit and explicit proportions became more coherent, such that a child's estimates of fractions on a 0–1 number-line was a linear function of the same child's estimates of equivalent integers. To further investigate whether preexisting integer knowledge can facilitate fraction learning through analogy, we assigned 100 third to fifth graders (M = 10.04 years, 55 girls, 76% White) to an Alignment condition, where children estimated fractions and integers on aligned number lines, or to a No Alignment condition (Study 2). Results showed that aligning integers and fractions on number lines facilitated a better understanding of fractional magnitudes, and increased children's fraction estimation accuracy to the level of college students'. Together, findings suggest that analogies can play an important role in building a coherent understanding of proportions.

Keywords: number line estimation, mathematical development, analogical reasoning, numerical cognition, progressive alignment

Understanding fractions is important for academic and professional success. In academic settings, fraction knowledge uniquely predicts performance in algebra and overall mathematics, both concurrently (Siegler et al., 2011) and longitudinally (Siegler et al., 2012). In professional settings, fraction knowledge continues to be essential. Over 65% of a nationally representative sample of adults in the United States reported using fractions at work (Handel, 2016). Despite their importance in school and the workplace, however, fractions are notoriously difficult to learn. Both children and even highly educated adults experience considerable challenges with fractions (Fazio et al., 2016; Ni & Zhou, 2005; Opfer & DeVries, 2008).

Despite the centrality of fraction understanding, there is little consensus regarding its development (Gallistel & Gelman, 1992; Geary, 2006; Siegler et al., 2011). Specifically, there is an ongoing debate on the role of integers in fraction learning, whether fraction understanding emerges independently from integer knowledge, and whether integer understanding supports or impedes fraction understanding.

In the present article, we investigate these issues by examining children's estimates of integers and fractions on number lines. Number lines are a valuable diagnostic tool because they require a continuous, scalar response to the implicit and explicit proportions denoted by integers and fractions. When estimating the position of the integer 3 on a 0–8 number line, for example, the proportion 3/8 is only implicit. In contrast, when estimating the fraction 3/8 on a 0–1 number line, the proportion 3/8 is explicit. Thus, comparing estimation of integers and fractions on number lines provides a robust test of the idea that fractions and integers develop separately versus jointly.

Based on previous results (Mack, 1995), we hypothesized that estimates of implicit proportions (i.e., estimating integers on number lines) and explicit proportions (i.e., estimating fractions on number lines) would be weakly correlated at the outset of learning, but would converge into a coherent, linear representation with

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schooling. Further, because ratios are the defining property of all rational numbers, we hypothesized that the coherence of fraction and integer estimation on number lines would predict the ability to deal with fractions in other, nonestimation tasks.

Separate Development of Integer and Fraction Understanding

Early research on fraction learning proposed that integer understanding was qualitatively different from, and even interfered with, fraction understanding. In line with these views, even after years of fraction instruction, children mistakenly generalize properties of integers to fractions (Ni & Zhou, 2005; Thompson & Opfer, 2008; Vamvakoussi & Vosniadou, 2010). Children are also often biased by the whole-number components of a fraction. In fraction comparison or ordering, for example, many children rely solely on either numerators or denominators—that is, incorrectly choosing the one with a larger numerator (or denominator) as a larger fraction, 4/12 > 2/3 (Hartnett & Gelman, 1998). Indeed, even college students are slower and less accurate when the smaller fraction has larger components, exhibiting inaccurate whole-number strategies (Fazio et al., 2016; Fitzsimmons et al., 2020b; Opfer & DeVries, 2008).

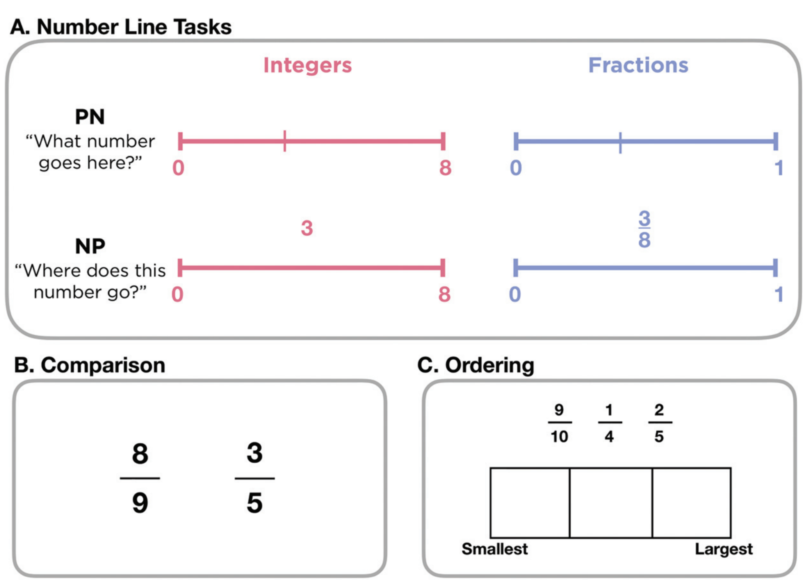
According to Gelman and others (Gallistel & Gelman, 1992), our innate number sense evolved to handle discrete quantities, thereby rendering the concept of fractional number counterintuitive. A similar argument is made from an evolutionary perspective (Geary, 2006), with integer processing depicted as humans' innate competence, and fractions depicted as the product of the culture in modern society, which must be learned laboriously (Dehaene, 2011; Feigenson et al., 2004). These perspectives echo Kronecker's claim that "God made the integers; all else is the work of man" (Bell, 2014).

Integrating Integer and Fraction Understanding via Analogical Bootstrapping

In contrast with this older research, a growing body of new research suggests that intuitive understanding of fractions emerges from infancy and even in nonhuman animals. For example, human infants are capable of distinguishing between proportions (e.g., 6:18 dots vs. 11:22 dots; McCrink & Wynn, 2007) and making probabilistic inferences from proportions (Denison & Xu, 2014). Beyond humans, nonhuman animals, such as newly hatched chicks and rhesus monkeys, can compare perceptual proportions (Rugani et al., 2016; Vallentin & Nieder, 2008). Therefore, it is possible that an innate ratio processing system, parallel with the innate system for processing integers, could support the development of fractional knowledge (Kalra et al., 2020; Lewis et al., 2016; Matthews & Chesney, 2015; Matthews et al., 2016). Moreover, there might also exist general systems that represent both integer and fraction magnitudes, such as a general magnitude system that represents both discrete and continuous quantities (Leibovich et al., 2017; Newcombe et al., 2015), or a general number system that represents all rational numbers (Clarke & Beck, 2021).

There is also evidence suggesting that representations of explicit proportions denoted by fractions are closely related to representations of implicit proportions denoted by relations between integers (Fazio et al., 2014; Iuculano & Butterworth, 2011; Jordan et al., 2013). In Fazio et al. (2014), for example, children were asked to estimate a given number (a large integer or a fraction) on a number line flanked by two numbers (Figure 1A). To-be-estimated integers and fractions were designed to be in approximately the same position on number lines (e.g., estimating '375' on a 0–1,000 number line or '3/8' on a 0–1 number line). The researchers found that children's estimates of large integers were more accurate and linear than their estimates of fractions. The accuracy of large-integer

Figure 1
Illustration of Tasks in Study 1



Note. PN = position-to-number task; NP = number-to-position task. See the online article for the color version of this figure.

estimates positively correlated with that of fraction estimates, suggesting that representations of large integers and fractions are related to each other.

In light of such considerations, Siegler et al. (2011) proposed an integrated theory of numerical development that emphasizes the developmental continuity between integers and fractions. According to the integrated theory, there are four important trends in numerical development: (1) accurately representing nonsymbolic numeric value (e.g., 3 dots); (2) associating symbolic numbers with nonsymbolic numeric values (linking '3' to 3 dots); (3) extending numerical understanding from small to large integers; and (4) extending integer understanding to understanding of rational numbers, including negatives, decimals, and fractions (Siegler & Lortie-Forgues, 2014). From this perspective, the cognitive system for representing integers is not an obstacle standing in the way of understanding fractions, but a stepping stone to understanding them all.

How we extend integer understanding to understanding other rational numbers, however, is unclear. We propose that analogy could serve as an underlying mechanism for this process. Generally, analogies allow learners to bootstrap learning far beyond what the input provides (Gentner, 1983). Indeed, analogical reasoning is widely used in our daily life and science (Hofstadter & Sander, 2013; Holyoak, 2012), and plays an important role in cognitive development (Gentner, 2010; Siegler, 1989).

At the same time, analogical reasoning often requires cognitive support, so merely knowing the meaning of integers would not be sufficient to understand fractions. One important cognitive support is direct alignment (i.e., spatially aligning matching components between targets and bases; Gentner, 1983; Gentner & Markman, 1997; Matlen et al., 2020). This alignment supports encoding of the maximal structural similarity between two representations; thereby, facilitating the abstraction of common relational structure and analogical transfer (Falkenhainer et al., 1989; Wolff & Gentner, 2000). Direct spatial alignment is also a characteristic of math education in East Asian schools, despite being less prominent in U.S. schools (Richland et al., 2007).

Analogies between well-understood small numbers and poorly understood fractions seem especially likely to be helpful. The reason is that integer understanding develops piecemeal (Siegler & Opfer, 2003; Thompson & Opfer, 2010), with the understanding of small integers (like 3 or 37) being superior to the understanding of large integers (like 375). Thus, children's estimates of 3 on a 0–8 number line would likely be more accurate than their estimates of 375 on a 0–1,000 number line or their estimates of 3/8 on a 0–1 number line.

Additionally, comparing the position of familiar and unfamiliar numbers on number lines does seem to elicit proportional analogies (Opfer & Martens, 2012; Opfer & Siegler, 2007). For example, Thompson and Opfer (2010) showed that by comparing the position of 15 on a 0–100 number line to that of 150 on a 0–1,000 number line, second graders who were typically accurate on 0–100 number lines and erroneous on 0–1,000 number lines quickly improved their estimates of numbers between 0 and 1,000. Likewise, proportional analogies (e.g., comparing the location of 3/8 on a 0–1 number line to the location of 3 on a 0–8 number line) may extend understanding of small integers to fractions.

These two ideas—that the proportional magnitudes of small integers are understood before the proportional magnitude of fractions and that comparison of small integers and fractions leads to a better understanding of fractions—have never been tested. If true, they would immediately suggest a novel educational intervention. In the current study, we aimed to conceptually replicate and extend Thompson and Opfer (2010) to examine whether we can use aligned number lines between integers and fractions to further extend people's small integer knowledge to fractions. More broadly, we hypothesized that analogy provides a way for children to develop representations of a potentially infinite number of numbers on a continuous number line when they can only have limited experiences with particular numbers.

Learning Fractions With Number Lines

Many studies have shown that depicting the position of fractions on number lines facilitates fraction understanding among both typically developing elementary students (Fazio et al., 2016; Moss & Case, 1999; Saxe et al., 2007) and at-risk math learners (Dyson et al., 2020; Fuchs et al., 2016, 2013, 2014). Fraction interventions emphasizing the measurement interpretation of fractions (i.e., fractions are numbers that are used to measure quantities; Hecht et al., 2003) led to better fraction proficiency than traditional curricula emphasizing the part-whole relation of numerators and denominators (Fuchs et al., 2014). This effect of fraction number-line intervention appears both broad and durable (Dyson et al., 2020).

Unfortunately, most previous interventions on fraction learning have included multiple evidence-based instructional components (e.g., graduated sequence, direct instruction, and strategy instruction; Hwang et al., 2019; Misquitta, 2011) over multiple sessions (e.g., 5–36 sessions with each session ranging from 30 min to 7 hr, and the majority of the studies implemented the intervention 2–4 days per week; Roesslein & Codding, 2019; Shin & Bryant, 2015). Thus, it is difficult to identify the "active ingredient" in facilitating a better representation of fractions. This is unfortunate because it leaves the mechanism of cognitive development unclear.

A few studies that have aimed to explain the effectiveness of number line interventions suggest that number lines are effective because they reduce numeric value to a single dimension of transitive relations (Fazio et al., 2016; Gunderson et al., 2019; Hamdan & Gunderson, 2017). For example, Gunderson and colleagues found that a fraction number-line training improved students' knowledge of fractional magnitudes more than an area-model training, suggesting that the unidimentionality of number lines is critical (Gunderson et al., 2019; Hamdan & Gunderson, 2017). The number line apparently highlights an important similarity between integers and fractions, that, just like integers, fractions can be ordered on a single dimension (Gunderson et al., 2019). Thus, a proportional relation between two components (i.e., the numerator and the denominator) of a fraction is reduced to a single point on a unidimensional line. As processing complex relations between fraction components requires working memory (Doumas et al., 2008; Halford et al., 1998; Hummel & Holyoak, 1997; Kalra et al., 2020), representing fraction magnitudes on a number line may free working memory resources for richer encoding and problem-solving (e.g., ordering multiple fractions by magnitude).

Another important feature of previous interventions is that all of the above-mentioned studies taught children to estimate fractions on number lines by providing extensive, item-by-item corrective feedback, thereby fostering development via reinforcement learning. An appealing feature of analogical learning is that it entails "zero-shot," unsupervised learning (Gentner, 2010; Kurtz et al., 2001; Opfer & Doumas, 2008; Siegler, 1989). From the perspective of the Knowledge-Learning-Instruction framework (Koedinger et al., 2012), learning events can be categorized into "memory and fluency processes" (effective for learning facts), "induction and refinement" (effective for learning rules), and "understanding and sense-making" (effective for learning principles). To further explore whether analogical bootstrapping helps children better understand fractions by learning the relational structure between integers and fractions, the current study focused more on students generating internal schema ("understanding and sense-making processes") rather than memorizing specific magnitudes reinforced by corrective feedback ("memory and fluency processes"). Whether children can learn fractional magnitudes without feedback is an intriguing possibility, especially given the inherent difficulty of learning fractions and the extensive feedback featured in previous studies (e.g., Fazio et al., 2016).

Current Studies

The current studies investigated (a) whether representations of integers and fractions develop independently or become more coherent over the course of development (Study 1) and (b) whether structurally aligning integers and fractions would facilitate understanding of one, both, or neither (Study 2).

In Study 1, we conducted a cross-sectional study examining third, fourth, and fifth graders' number-line estimates of integers and fractions that were proportionally identical to the upper bound of a number line (e.g., 3 on a 0–8 number line or 3/8 on a 0–1 number line). These grades are particularly important to test because instruction on fractions begins in the middle of third grade (i.e., after our testing) and continues through fifth grade (Common Core State Standards Initiative, 2010). By focusing on this grade range, we could examine what children might typically learn at school during this time. At the same time, the instruction does not (yet) include a systematic comparison of the placement of integers and fractions on number lines, so it is not clear whether or when children might treat them as proportionally equivalent on number lines (see Figure 1A).

We expected that if integer and fraction representations develop independently (Gallistel & Gelman, 1992; Geary, 2006), estimates for proportionally equivalent integers and fractions would be statistically independent (i.e., not reliably correlated). In contrast, if children develop a coherent representation of fractional and integer magnitude, estimates of fractions and integers would be strongly related to each other, with correlations strengthening and the discrepancy between the proportions estimated for integers and fractions declining over time. To test this hypothesis, we measured representational coherence between integers and fractions using two novel measures: (a) the correlation between integer and fraction estimates on proportionally equivalent number lines (Integer-Fraction Correlation, IFC) and (b) the absolute difference between estimations of integers and fractions (Integer-Fraction Discrepancy, IFD), which we will detail in the Method section.

In Study 2, we investigated whether analogies to implicit proportions denoted by relations of integers would facilitate understanding of explicit proportions denoted by fractions. Using a pretest-training-posttest design, we randomly assigned third-to-fifth graders to one of two conditions that differed only in the training phase. In the Alignment condition, children solved aligned pairs of number-line problems; in the No Alignment condition, children solved the same problems one at a time.

We expected that preexisting knowledge of integers would help children better understand the nature and structure of fractions given cognitive support of alignment. We also explored whether comparing integers to fractions would improve judgments of integers. An interesting feature of analogical bootstrapping is that comparison of two partially understood domains leads to improvements in both domains, not just the poorly understood domain (Kurtz et al., 2001).

Study 1

Method

Participants

One hundred and seventy-seven third-to-fifth graders (69% White; 17% low-income; 54 third graders, *M* = 8.79, *SD* = .31, 26 females; 66 fourth graders, *M* = 9.90, *SD* = .42, 30 females; and 57 fifth graders, *M* = 10.80, *SD* = .40, 31 females) from two public school districts in the Midwestern United States participated in the current study at the beginning of the fall semester. This study was approved by the Institutional Review Board (IRB) of the Ohio State University (Project 2013B0450: Early Development in Mathematical Skills).

Materials and Procedure

Participants completed the following six tasks on a laptop in a random order: four number-line tasks (2 number formats [integer or fraction] × 2 tasks [position-to-number or number-to-position]), fraction comparison, and fraction ordering (see Figure 1). Data and study materials for Study 1 and Study 2 are available at Open Science Framework (OSF, https://osf.io/cetjp/?view_only=116ad3d9bf454e568cf59a248300f1ca; Yu et al., 2021). The studies were not preregistered.

Fraction Position-to-Number (FPN) Task

In the Fraction Position-to-Number task, children needed to produce a number corresponding to a mark on a 0–1 number line (Iuculano & Butterworth, 2011). First, children saw a number line flanked with 0 and 1, and were instructed "*In this game, you are going to see a number line like this. Each number line will have a 0 at this end and a 1 at the other end. There is also going to be a hatch mark somewhere on the line. Your job is to estimate what fraction goes with the mark.*" Then, a total of 10 number lines were presented sequentially, with "0" marked on the left end and "1" marked on the right end. On each trial, children were presented with a location indicated by a hatch mark on the number line and were asked to estimate which fraction corresponded to the mark. To-be-estimated fractions were 1/11, 1/7, 1/4, 3/8, 2/5, 4/7, 2/3, 7/9, 5/6, and 9/10. These magnitudes were chosen to be evenly distributed on the number line, with one fraction drawn from each tenth of the number line.

Integer Position-to-Number (IPN) Task

The integer PN task was identical to the fraction PN task, except for the to-be-estimated and right-end numbers. The to-be-estimated integers and right-end numbers were chosen based on the fractions included in the fraction PN task, such that stimuli were proportionally identical on the number lines to those in the fraction task. For example, for the fraction 3/8, the corresponding integer number line ranged from 0 to 8, and the mark was located at the position of 3. We kept integers in this task the same as the integer components of fractions rather than using a fixed right-end number (e.g., 1,000 in Fazio et al., 2014) to control the size of numbers. Ten integers that corresponded to fractions used in the fraction PN task were used: 1 on a 0–11 line, 1 on a 0–7 line, 1 on a 0–4 line, and so on. At the start of the task, children saw a number line with "0" marked on the left end and "?" marked on the right end. Children were instructed "*In this game, you are going to see a number line like this. Each number line will have a 0 at this end and a whole number at the other end. There is also going to be a hatch mark somewhere on the line. Your job is to estimate what number goes with the mark. You need to pay attention to this number (pointing to the right end) because it may change.*" On each trial, children were asked to estimate which integer corresponded to the mark.

Fraction Number-to-Position (FNP) Task

In this typical fraction estimation task (e.g., Siegler et al., 2011), children estimated the position of a given fraction on a number line flanked by "0" and "1" by dragging a hatch mark. At the start of each trial, the hatch mark was located at 0. To-be-estimated fractions were the same magnitudes used in the fraction PN task. Additionally, to investigate whether children's estimates of equivalent fractions are affected by the size of individual components of fractions (i.e., numerators and denominators), children also estimated equivalent fractions with independent components that had been doubled (2/22, 2/14, 6/16, etc.; referred to as *large-component fractions*; Braithwaite & Siegler, 2018; Fitzsimmons & Thompson, 2022; Fitzsimmons et al., 2020a, 2020b), resulting in 20 trials in total.

Integer Number-to-Position (INP) Task

In this task, children estimated the location of an integer on the number line by dragging a hatch mark. This task differed from typical number line estimation tasks (Siegler & Opfer, 2003) in that the right endpoint differed from trial to trial. As in the integer position-to-number task, children were told that the right endpoint might change in the instructions. Children completed a total of 10 trials in which the magnitudes were the same as those used in the integer PN task.

Comparison

The comparison task was the same as in Siegler et al. (2011). In this task, children compared 3/5 to each of the following fractions: 2/9, 3/8, 5/9, 4/7, 5/8, 2/3, 4/5, and 8/9. These fractions were chosen so that there were equal numbers of magnitudes to be compared on both sides of 3/5. On each trial, we asked children to press the left or right arrow on the keyboard to choose the larger of two fractions (e.g., 3/5 and 2/9). Fraction pairs remained on the screen until a response was made. For each comparison pair, the larger fraction would appear on the left once and on the right once, resulting in a total of 16 trials.

Ordering

The ordering task was a simplified version of Mazzocco and Devlin (2008). On each trial, children were asked to order three fractions from smallest to largest. To-be-ordered fractions were chosen from the following fractions: 1/7, 1/4, 3/8, 2/5, 4/7, 2/3, 7/9, 5/6, and 9/10. There were nine trials in total, with each of the nine to-be-ordered fractions appearing in 3 different trials.

Results

We first present descriptive results for comparison, ordering, and number-line tasks. These tasks all had high internal reliability, with Cronbach's alpha ranging from .76–.94 (comparison: .79, ordering: .92, integer number-to-position: .82, integer position-to-number: .76, fraction number-to-position: .84, fraction position-to-number: .94). Then, we used two representational-coherence measures: (a) Integer-Fraction Correlation and (b) Integer-Fraction Discrepancy (see below for details), to examine the coherence between integer and fraction representations, how this coherence changed developmentally, and whether this coherence predicted accuracy of comparison and ordering.

Comparison

The average accuracy increased with grade (49.9%, 63.3%, and 79.2% for third, fourth, and fifth graders, respectively). Accuracy on each trial of the comparison task was analyzed using a generalized mixed-effects model with by-participant and by-item random intercepts, and grade (coded as a continuous variable throughout the article) as fixed effects.[1] The analysis was conducted using the *lme4* package (Bates et al., 2015) in *R* (Version 3.6.3; R Core Team, 2020). The likelihood of comparing fractions correctly increased as grade increased, $\beta = .70$, $SE = .09$, $p < .001$.

Ordering

We scored each trial based on how many fraction pairs children ordered correctly within each triplet, ranging from 0 to 3, and summed a total score for each participant. Accuracy was calculated as the percent correct out of 27 (i.e., nine trials with a possible score of three for each trial) for each participant. Average accuracy for third, fourth, and fifth graders was 81.2%, 81.0%, and 84.3%, respectively. Scores on each trial were analyzed using a linear mixed-effects model with by-participant and by-item random intercepts, and grade as a fixed effect. There was not a significant effect of grade on scores on the ordering task, $\beta = .04$, $SE = .06$, $p = .434$.

Number Line Estimation

To assess children's accuracy in number line estimation tasks, we calculated percent absolute error (PAE) for each trial, which is defined as |to-be-estimated number - child's estimate|/ number range. For example, if a child estimated the given fraction of 3/8 as 2/5 on a 0–1 number line, then PAE = $(|3/8 - 2/5|)/1 \times 100\% = 2.5\%$. In the PN tasks, children identified which number was represented by a given hatch mark on a number line. Thus, there were responses that were larger than the upper bounds (e.g., 3/2 for the given magnitude of 3/8 on a 0–1 number line). To exclude these extreme values, PAE larger than 3 *SD*s of the mean were excluded from further analyses (1% of the data).

As expected, the average PAE for fractions (PN: 54.3%, NP: 16.7%) was greater than those for integers (PN: 9.8%, NP: 9.2%).

[1] Results remained similar when we coded grade as a categorical variable.

Children were more accurate (i.e., lower PAE) in their estimates of integers than fractions in both tasks and became more accurate in both tasks and number formats across grades (Figure 2A). We examined the effects of grade, number format, and task in a linear mixed-effects regression with grade, number format (integer, fraction), and task (PN, NP) as fixed effects, with by-participant random slopes for number format and task, by-participant random intercepts, and by-item random intercepts. The distribution of PAE was skewed, so we used log-transformed PAE as the dependent variable (also see Marchand et al., 2020). Because the fraction NP task involved 10 additional large component-size fractions which did not overlap with the other three number line estimation tasks, we only included trials with overlapping to-be-estimated numbers across the four tasks (i.e., fractions with small independent components) in this model to disentangle the effect of component size and the effect of tasks.

Results revealed that PAE of children's estimates decreased (i.e., estimates became more accurate) as grade increased (third: 34.0%, fourth: 18.7%, fifth: 16.0%), $\beta = -.23$, $SE = .03$, $p < .001$. As predicted, children were more accurate in their estimates of integers than fractions (integer PAE: 9.5%, fraction PAE: 35.5%), $\beta = -.35$, $SE = .02$, $p < .001$, and they were more accurate on the PN task than the NP task (PN PAE: 32.2%, NP PAE: 13.0%), $\beta = -.08$, $SE = .02$, $p < .001$. There was also a Task × Format interaction, $\beta = .18$, $SE = .01$, $p < .001$, which indicated a larger advantage for the integer format over fraction format in the PN task ($\beta = .56$, $SE = .01$) than the NP task ($\beta = .15$, $SE = .01$). In addition, we also observed a Grade × Task interaction, $\beta = -.06$, $SE = .02$, $p < .001$, indicated by a steeper decrease in PAE with grade in the PN task ($\beta = -.30$, $SE = .03$) than the NP task ($\beta = -.17$, $SE = .03$). More importantly, there was a Grade × Format interaction, $\beta = -.07$, $SE = .02$, $p < .001$, indicated by a greater decrease in PAE with grade for fractions ($\beta = -.30$, $SE = -.03$) than integers ($\beta = -.17$, $SE = .03$). In other words, the advantage of the integer format over fraction format reduced as grade increased. This finding suggests that children may be developing a more coherent representation of integers and fractions, an issue we examine more directly in the next section.

To examine the effect of component size on estimation accuracy, we submitted PAE on the fraction NP task to a linear mixed-effects model with grade and component size (small-component fraction, e.g., 3/8, vs. large-component fraction, e.g., 6/16) as fixed effects, by-participant random slopes for component size, by-participant random intercepts, and by-item random intercepts. We again found that children were more accurate in their estimates as grade increased, $\beta = -.28$, $SE = .04$, $p < .001$. Children's PAE was similar for large-component and small-component fractions, $\beta = .02$, $SE = .05$, $p = .653$. However, grade and component size had small interactive effects, $\beta = .04$, $SE = .01$, $p < .01$, indicating that the advantage for small components increased with grade (third: large-component 23.0%, small-component 25.0%; fourth: large-component 16.7%, small-component 15.9%; fifth: large-component 10.7%, small-component 10.0%).

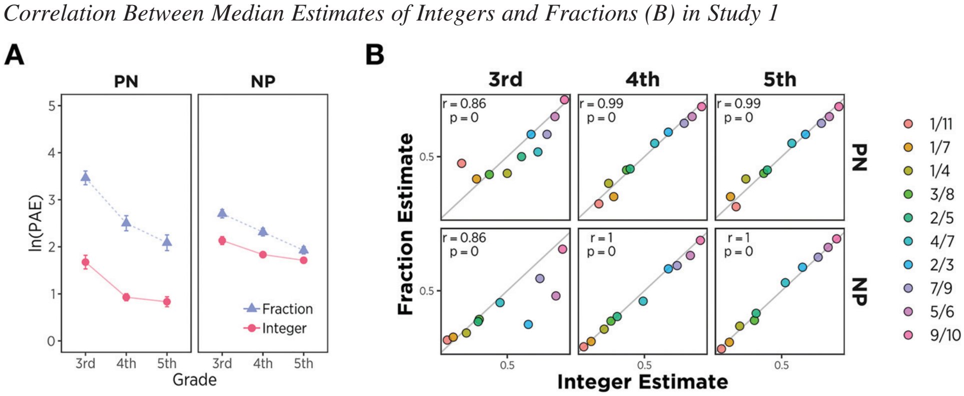
Developing Coherent Integer and Fraction Representations

To investigate representational coherence, we used two representational-coherence measures: (a) Integer-Fraction Correlation, which is the correlation between estimates of proportionally equivalent integers and fractions (e.g., 3 on a 0–8 and 3/8 on a 0–1 number line; see simulations in Figure 3); and (b) Integer-Fraction Discrepancy, which is the absolute difference between estimates of proportionally equivalent integers and fractions (e.g., |(estimate of 3)/(upper bound of 8) – (estimate of 3/8)|). For example, if a participant places five to the question "Where is 3 on a 0–8 number line?" and places 5/8 to the question "Where is 3/8 on a 0–1 number line?," then the Integer-Fraction Discrepancy would be 5/8 – 5/8 = 0, indicating coherent (but inaccurate) representations of integers and fractions. Because estimates can be larger than the upper bound, trials with IFD beyond 3 *SD*s of the mean were excluded from further analyses on IFD (.23% of total trials).

To further illustrate how the IFC varies when participants have incoherent representations of integers and fractions, we ran simulations of IFC generated from different combinations of compressive, expansive, and linear representations of integers and fractions (see Figure 3). More specifically, we simulated mean representations of integer and fraction magnitudes by the equation: $Estimates = a \times GivenProportion^{\beta}$. a is the scaling parameter. β indicates how compressive or expansive the representation is, with $\beta < 1$ as compressive representations, $\beta = 1$ as linear representations, and $\beta > 1$ as expansive representations. In our simulations, for the compressive

Figure 2

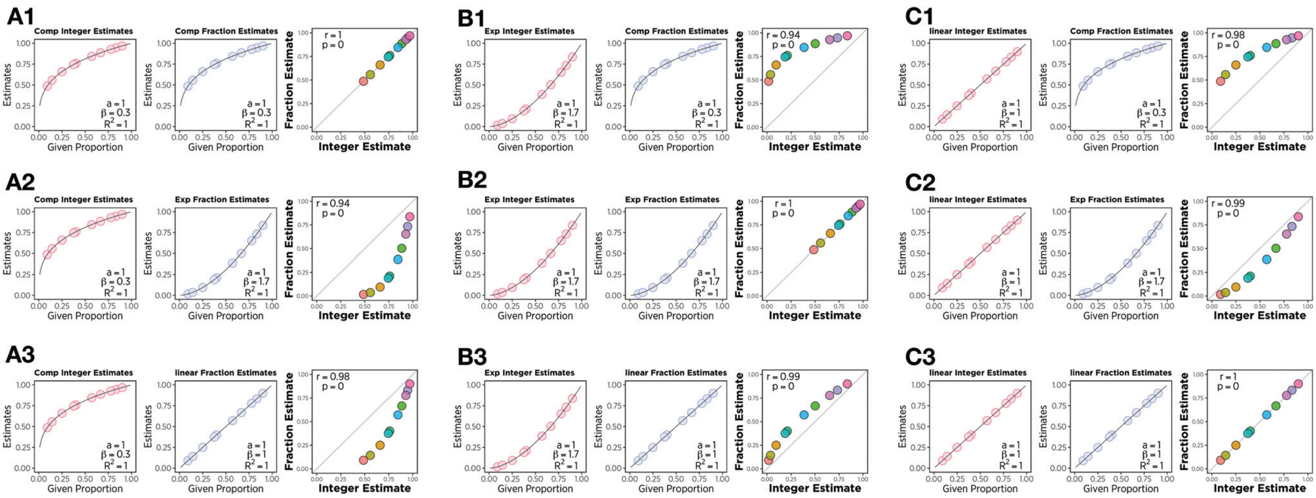
Percent Absolute Error (PAE) of Fraction and Integer Estimates (A) and Integer-Fraction Correlation Between Median Estimates of Integers and Fractions (B) in Study 1



Note. Error bars indicate ± *SEM*. PN = position-to-number task; NP = number-to-position task. See the online article for the color version of this figure.

Figure 3

Simulations of Integer-Fraction Correlation Generated From Different Combinations of Compressive, Expansive, and Linear Representations of Integers and Fractions



Note. The simulation equation is $Estimates = a \times GivenProportion^{\beta}$. a is the scaling parameter. β indicates how compressive or expansive the representation is, with $\beta < 1$ as compressive representations, $\beta = 1$ as linear representations, and $\beta > 1$ as expansive representations. In our simulations, for the compressive representations, $a = 1$, $\beta = .3$; for the expansive representations, $a = 1$, $\beta = 1.7$; and for the linear representations, $a = 1$, $\beta = 1$. For example, A2 shows that the Integer-Fraction Correlation is .94 when a participant has a compressive representation of integers and an expansive representation of fractions. Exp = expansive; Comp = compressive. See the online article for the color version of this figure.

representations, $a = 1$, $\beta = .3$; for the expansive representations, $a = 1$, $\beta = 1.7$; and for the linear representations, $a = 1$, $\beta = 1$. For each combination, Figure 3 plots representations of integer and fraction magnitudes as a function of given proportions (e.g., to-be-estimated magnitudes used in our number line estimation tasks) and the best-fitting functions. Our simulations showed that IFC is the highest when representations of integers and fractions follow the same function, even when the best-fitting function is not necessarily the most accurate one.

As these simulations make clear, neither the IFC nor the IFD is simply accuracy. For example, if both fraction and integer estimates increased logarithmically (or even sinusoidally), estimates would be inaccurate but highly correlated (Figure 3 A1). Or, if both fraction and integer estimates increased linearly, but the slope of the functions differed, there would be a high correlation but also a high discrepancy. Conversely, if estimates followed a cyclic power function—estimates increased as a power function of true values that are biased by reference points such as ends and the middle of the number line (Hollands & Dyre, 2000; Slusser et al., 2013)—but the two functions were out of phase (e.g., children use a different number of reference points for integer and fraction estimation two estimation functions involve different cycles), accuracy could be quite high, yet the discrepancy might also be high. Moreover, IFC is an integrated measure across all given proportions and IFD is the distance between each estimate of proportionally equivalent integers and fractions. Thus, the IFC and IFD measures provide a more direct and valid measure of representational coherence than mere accuracy.

Integer-Fraction Correlation

First, we computed IFC with overall median estimates. IFC was strong from the third grade (PN: $r = .86$, $p < .01$; NP: $r = .86$, $p < .01$) and became perfect in fourth and fifth grades in both PN and NP tasks (PN: $r = .99$, $p < .001$; NP: $r = 1.00$, $p < .001$; Figure 2B). The increase in correlation strength with age suggests that children are forming more coherent representations between different formats of numbers throughout development.

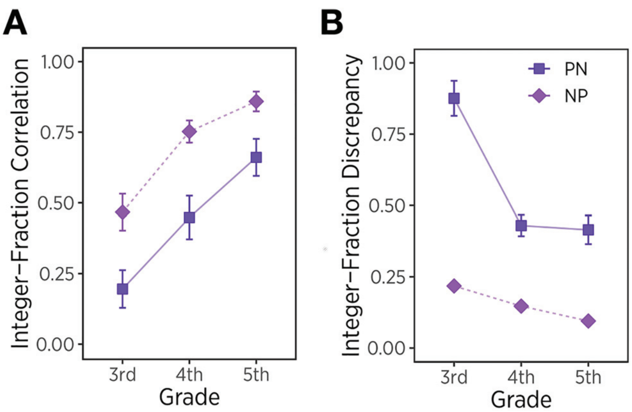
Then we computed IFC for each individual and submitted these values to a linear mixed-effects model with grade and task (PN, NP) as fixed effects and by-participant random intercepts. As shown in Figure 4A, IFC was stronger in the NP tasks than PN tasks, $\beta = .26$, $SE = .04$, $p < .001$. There was no significant Task × Grade interaction, $\beta = .03$, $SE = .04$, $p = .422$. IFC increased significantly as grade increased, $\beta = .34$, $SE = .06$, $p < .001$.

Integer-Fraction Discrepancy

We next conducted identical analyses on IFD with overall median estimates and individual data. On median estimates, IFD decreased with age (PN:.09,.03,.03 for third, fourth, and fifth grade; NP: .12, .02, .02 for third, fourth, and fifth grade, respectively).

Then, we ran a linear mixed-effects model on individual-level IFD values with grade and task (PN, NP) as fixed effects, by-participants random slopes for task, and by-participants random intercepts. IFD was larger in the PN tasks than NP tasks, $\beta = .24$, $SE = .04$, $p < .001$. Consistent with the findings with IFC, IFD decreased significantly as grade increased, $\beta = -.13$, $SE = .04$, $p < .001$, indicating children develop a more coherent representation of integers and fractions with age and education (Figure 4B). There was also a significant interaction effect between grade and task, $\beta = -.08$, $SE = .04$, $p < .05$, indicated by a larger decrease of IFD on the PN task with grade compared with the NP task (PN: .88, .43, .41 for third, fourth, and fifth grade; NP: .22, .15, .09 for third, fourth, and fifth grade, respectively).

Figure 4
Mean Integer-Fraction Correlation (A) and Integer-Fraction Discrepancy (B) in Study 1



Note. Error bars indicate ± *SEM*. PN = position-to-number task; NP = number-to-position task. See the online article for the color version of this figure.

Relations Between Representational Coherence and Fractional Proficiency

How does representational coherence relate to fraction understanding? To investigate this, we examined whether IFC and IFD scores on number-line tasks predicted accuracy of fraction comparison and ordering. Performance on fraction comparison showed substantial, positive associations with IFC scores (PN: $r = .48$, $p < .001$; NP: $r = .48$, $p < .001$) and negative associations with IFD scores (PN: $r = -.30$, $p < .01$; NP: $r = -.56$, $p < .001$; Table 1). Children's ordering performance was also correlated with both measures (IFC on the PN: $r = .19$, $p < .05$; IFC on the NP: $r = .21$, $p < .05$; IFD on the PN: $r = -.05$, $p = .515$; IFD on the NP: $r = -.26$, $p < .01$).

To check that these correlations did not arise simply because everything improves with age, we ran a generalized mixed-effects model with accuracy in fraction comparison on each trial as the dependent variable, by-participant random intercepts, and grade and IFC in the NP and PN tasks as fixed effects. Even after controlling for grade, we found that IFC significantly predicted comparison accuracy (IFC on the PN: $\beta = .29$, $SE = .08$, $p < .001$; NP: $\beta = .22$, $SE = .08$, $p < .01$), indicating that children with more coherent representations of integers and fractions made more accurate comparisons of fractions. When the same analysis was conducted on ordering accuracy, IFC on the NP task marginally predicted performance in the ordering task ($\beta = .14$, $SE = .07$, $p = .053$). Thus, coherence of fraction and integer estimates on the number-line task predicts a better ability to compare and order fractions.

We then carried out the same analysis with IFD. For the comparison task, IFD negatively predicted accuracy (IFD on the PN: $\beta = -.15$, $SE = .07$, $p < .05$; NP: $\beta = -.43$, $SE = .08$, $p < .001$). For the ordering task, IFD on the NP task negatively predicted accuracy ($\beta = -.23$, $SE = .06$, $p < .001$). Thus, having a less discrepant representation of integers and fractions predicts a better ability to compare and order fractions.

Discussion

Understanding the development of integer and fraction representations is important for mathematical cognition and education. Here, we examined the developmental trajectory of integer and fraction estimates and found that integer and fraction representations become more coherent with age, and that this representational coherence independently predicts fraction proficiency.

Consistent with previous research (Fazio et al., 2014), we found that children's estimates of integers and fractions were positively related, and both became more accurate with age. Extending the integrated theory (Siegler et al., 2011), we demonstrated that the development of integer and fraction understanding is not independent, but related to one another: children acquired more coherent representations for integers (implicit proportion) and fractions (explicit proportion) over development. Furthermore, individual differences in representational coherence predicted fraction proficiency: children with more coherent representations between integers and fractions were more accurate in fraction comparison and ordering tasks.

The findings shed light on the importance of building a coherent representation of numbers in education, but what leads to discrepant representations of integers and fractions and how can researchers and teachers facilitate integrated representations? One conceptual barrier to learning fractions may be a lack of understanding of the measurement properties of fractions (i.e., fractions denote magnitudes, and their magnitudes can be represented as positions on number lines; Hecht et al., 2003). Thus, the integer components of fractions, that is, numerators and denominators, can bias a holistic representation of fractional magnitudes (Ni & Zhou, 2005). A second barrier may be incorrect analogies between integer and fraction properties. Children may overgeneralize properties of integers to fractions, for example,

Table 1
Correlations Among Accuracy in Comparison and Ordering Tasks and Representational Coherence in Number Line Tasks in Study 1

Measures	Comparison	Ordering	PN IFC	NP IFC	PN IFD	NP IFD
Comparison	**.79****	.31*	.48*	.48*	−.30**	−.56*
Ordering		**.92*****	.19*	.21*	−.05	−.26**
PN IFC			**.71*****	.49*	−.31*	−.55*
NP IFC				**.61*****	−.23*	−.84*
PN IFD					**.81*****	.24*
NP IFD						**.72*****

Note. PN = position-to-number task; NP = number-to-position task; IFC = Integer-Fraction Correlation; IFD = Integer-Fraction Discrepancy. The statistics in bold on the diagonal are the internal reliability of each measure.
* $p < .05$. ** $p < .01$. *** $p < .001$.

integers have unique successors, and there is a finite number of integers between two integers (Vamvakoussi & Vosniadou, 2004, 2010).

We theorized that analogy serves as a learning mechanism that helps children extend their mental number line from familiar numbers to novel ones. An analogy between integers and fractions would be a special case of this general principle. If this is right, then setting up an experimental intervention that provides cognitive support (alignment) to facilitate analogy would be expected to lead to changes that are normally seen in development. To test this hypothesis, in Study 2, we designed a training study to examine whether alignment of fractions to integers on equivalent number lines would facilitate a better understanding of fractional magnitudes.

Study 2

Method

Participants

One hundred third-to-fifth graders (M = 10.04, SD = .83, 55% females, 76% White; 25 third graders, M = 9.07, SD = .45, 52% females; 42 fourth graders, M = 9.94, SD = .47, 62% females; and 33 fifth graders, M = 10.91, SD = .43, 48% females) participated in the study. We recruited participants from two public school districts in the Midwestern United States (60% of our sample) and through online advertisements posted at childrenhelpingscience.com (20%; Sheskin et al., 2020) and Facebook.com (20%). An additional 43 participants who quit before completing the training phase were excluded from further analyses (19 participants quit during the pretest, 24 participants quit during the training, $N_{Alignment}$ = 15, $N_{NoAlignment}$ = 9).

We also recruited 46 college students (M = 19.42, SD = 1.28, 43% females, 50% White) to establish a comparison level of accuracy in evaluating our training. Ideally, training would bring children to college student levels of accuracy. Adult participants received course credit for their participation. This study was approved by the IRB of the Ohio State University (Project 2013B0450: Early Development in Mathematical Skills).

Materials and Procedure

The current study was presented in Qualtrics and hosted online on http://discoveriesonline.org (Rhodes et al., 2020). Data was collected during the pandemic from May 2020 to February 2021 when a majority of school lessons were transferred to an online setting; thus, it was critical to examine the effect of online teaching. Families participated independently through a link at their preferred place and time remotely. Children watched prerecorded video instructions and completed the tasks online at their own pace, without direct interaction with experimenters. To make sure children saw complete task instructions, children could only advance the program after the video instructions were played. To minimize parental interference, before the study started, we asked parents to ensure that their children participated in a quiet environment and to provide no help to their children as they solved problems. Moreover, we video-recorded children's responses through the web camera to monitor potential interference. This unmoderated remote research setting minimizes interference from researchers and ensures that all instructions are standardized (Rhodes et al., 2020). Furthermore, our training protocol can be easily applied to classroom settings and scaled up to a larger sample of participants.

Child participants completed a pretest, training, and posttest phase in one session, with an average of approximately 50 min to complete the whole session. Adult participants only completed the pretest (see Figure 5).

Pretest and Posttest Battery

At pretest and posttest, children completed a simplified testing battery from Study 1, consisting of number-line estimation tasks: fraction position-to-number task, integer position-to-number task, fraction number-to-position task, and integer number-to-position task. The four estimation tasks and items within each task were presented in a randomized order.

The number-line estimation tasks were the same as Study 1, except that in the fraction NP task, only small-component fractions were included because we did not observe a strong effect of component size in Study 1. Therefore, there were 10 trials in each task.

Training Conditions

Children were randomly assigned to one of two conditions during training: Alignment (N = 49) or No Alignment (N = 51).

In the Alignment condition, we presented children with integer and fraction number lines that were aligned vertically on one screen (Figure 6A). We aligned the two number lines in this way

Figure 5

An Illustration of the Experimental Procedure in Study 2

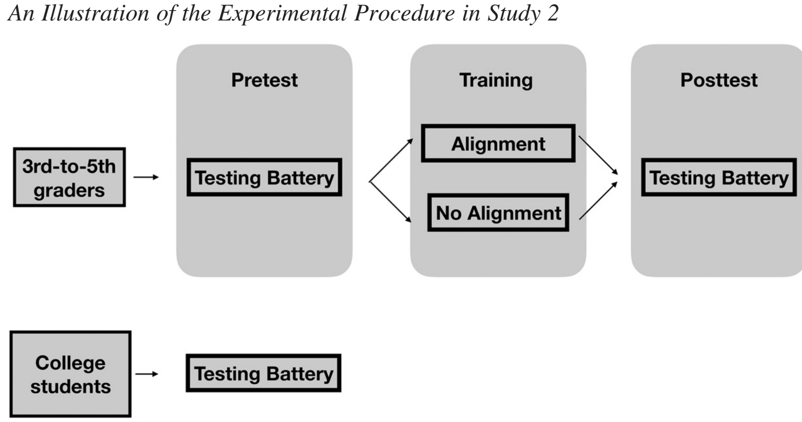
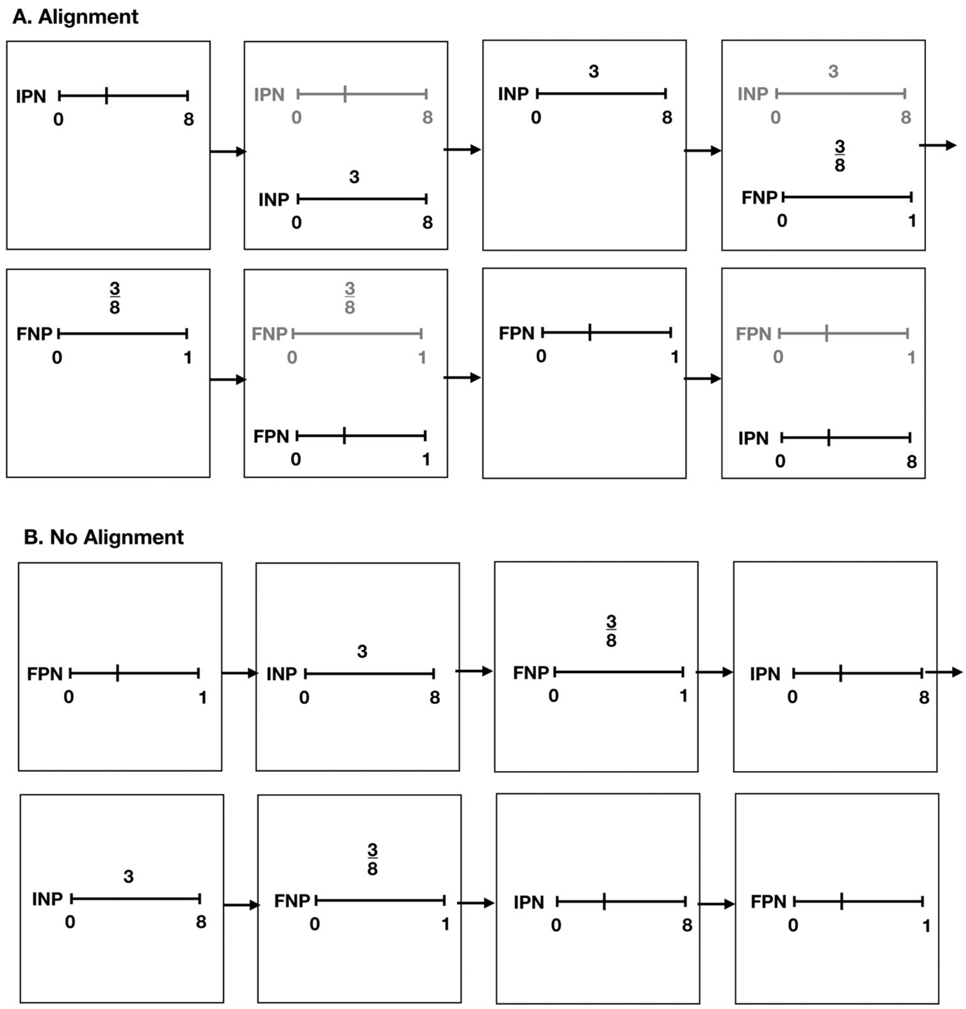


Figure 6
An Illustration of the Training Trials Across Conditions in Study 2



Note. Each black square denotes a trial. For each trained magnitude, in the Alignment condition, children solved two vertically aligned problems on each screen sequentially, with the top problem remaining on the screen when they solved the bottom problem. In the No Alignment condition, children solved the same problems one at a time on each screen in a randomized order; thereby, blocking the opportunity of visual alignment. IPN = integer position-to-number task; INP = integer number-to-position task; FPN = fraction position-to-number task; FNP = fraction number-to-position.

to highlight the structural similarity between integers and fractions. Participants were first asked to estimate the fraction (or equivalent) of 1/11 across eight NP and PN problems, and then the magnitude of stimuli increased as the training session progressed: 1/7, 1/4, 3/8, 2/5, 4/7, 2/3, 7/9, 5/6, and 9/10. In total, there were 80 problems (eight problems for each of 10 fractions) in the training session.

More specifically, for each fraction, children started with an integer PN problem at the top part of the screen. We chose to start with this task because Study 1 suggested that integer PN problems were the easiest for children out of the four number line estimation tasks. After children finished this problem, an integer NP problem appeared on the bottom part of the screen vertically aligned with the top problem, with the top problem still appearing on the screen. No feedback was given on any of the problems.

Then on the next screen, children completed the same integer NP problem on the top part of the screen. After that, a fraction NP problem appeared on the bottom part of the screen, so that children had an opportunity to compare fraction estimates with integer estimates on corresponding scales.

After that, on the next screen, children again completed the same fraction NP problem as they just did, but it was now located at the top of the screen, and a fraction PN problem located at the bottom of the screen, one at a time. Finally, on the next screen, children completed a fraction PN problem and a vertically aligned integer PN problem.

In the No Alignment condition, children completed the same items as in the Alignment condition with two exceptions: (a) only one problem was presented on the screen at a time, and (b) problems were presented in a randomized order across all trained magnitudes (Figure 6B); thereby, blocking the opportunity of visual alignment. In neither condition were children instructed to compare the problems, nor did they receive any feedback on their answers.

Video Coding

We used video recordings to check data validity (e.g., monitoring potential parental inference). About half of our sample successfully uploaded their videos (N = 46, 46% of the sample). Consistent with previous studies using online unmoderated research (e.g., Leshin et al., 2021), the prevalence of parental interference was small (fewer than 1% of all trials). Trials with interference were excluded from further analyses.

Results

To assess participants' performance on the number-line estimation tasks, we calculated PAE for each trial. Estimates larger than 1,000 on PN tasks were deleted from further analyses (less than .01% of the total trials). As in Study 1, there was a nonnormal distribution of PAE, so we again conducted analyses on log-transformed PAE. As in Study 1, PAE beyond 3 *SD*s from the mean was excluded from further analyses (.22% of the total trials). We also calculated the same measures of representational coherence as in Study 1, IFC and IFD. IFD beyond 3 *SD*s from the mean was excluded from further analyses (1.25% of the total trials).

In the following sections, we analyzed participants' estimation accuracy and their representational coherence during training and posttest while controlling for their performance at pretest, to investigate the effect of alignment on performance.

Training

First, to examine the effects of training on estimation accuracy, we ran a series of linear mixed-effects models on separate tasks with log-transformed PAE on each trial during training as the dependent variable, grade, pretest log-transformed PAE, and condition (Alignment, No Alignment) as fixed effects, by-participant random slopes for pretest performance, by-participant random intercepts, and by-item random intercepts.[2]

We found that children who were more accurate when estimating integers and fractions on the pretest were also more accurate in their estimates during training (IPN: $\beta = .25$, $SE = .04$, $p < .001$; INP: $\beta = .20$, $SE = .03$, $p < .001$; FPN: $\beta = .23$, $SE = .04$, $p < .001$; FNP: $\beta = .23$, $SE = .03$, $p < .001$). Consistent with our hypothesis, children in the Alignment condition were significantly more accurate (i.e., PAE was lower) with fraction estimates than children in the No Alignment condition on both PN and NP tasks (FPN: Alignment PAE M = 14.15%, $SD = .43$; No Alignment PAE M = 21.85%, $SD = .30$; $\beta = .61$, $SE = .09$, $p < .001$; FNP: Alignment PAE M = 5.47%, $SD = .06$; No Alignment PAE M = 10.46%, $SD = .11$; $\beta = .39$ $SE = .09$, $p < .001$; Table 2). What is more, alignment also yielded more accurate estimates of integers on the NP task (INP: Alignment PAE M = 3.96%, $SD = .04$; No Alignment PAE M = 5.46%, $SD = .03$; $\beta = .37$, $SE = .07$, $p < .001$) but not on the PN task (IPN: Alignment PAE M = 4.08%, $SD = .09$; No Alignment PAE M = 3.43%, $SD = .05$; $\beta = -.03$, $SE = .08$, $p = .711$). These results indicate that aligning fractions to integer number lines helped children more accurately estimate both fractional and integer magnitudes, even though children were not explicitly instructed to draw analogies.

Next, we compared children's estimation accuracy with that of college students to further examine the effect of alignment on estimation accuracy. To do this, we conducted a series of linear mixed-effects models on different tasks with PAE as the dependent variable, group (adults as the reference group, children in the Alignment group, and children in the No Alignment group) as a fixed effect, by-participant random intercepts, and by-item random intercepts. Results showed that when children viewed integer number lines aligned with fraction number lines, their fraction estimates were as accurate as, or even more accurate than, college students (Figure 7; FNP: adults PAE M = 4.40%, $SD = .04$, $\beta = -.07$, $SE = .12$, $p = .548$; FPN: adults PAE M = 3.58%, $SD = .02$, $\beta = -.30$, $SE = .13$, $p < .05$). However, the performance of children in the No Alignment condition were less accurate than college students (FNP: $\beta = .45$, $SE = .12$, $p < .001$; FPN: $\beta = .53$, $SE = .13$, $p < .001$).

Similarly, children in the Alignment condition were as accurate as college students when estimating integers (INP: adults PAE M = 3.79%, $SD = .02$, $\beta = -.18$, $SE = .10$, $p = .077$; IPN: adults PAE M = 1.50%, $SD = .03$, $\beta = .20$, $SE = .11$, $p = .076$). Children in the No Alignment condition did not differ from college students on the integer PN task (IPN: $\beta = .21$, $SE = .11$, $p = .058$), but they were less accurate than college students on the integer NP task (INP: $\beta = .25$, $SE = .10$, $p < .05$). Thus, although our intended subject of training had been fractions, alignment led to improved integer understanding as well.

Then we examined whether alignment facilitate representational coherence between integers and fractions. We first conducted a linear mixed-effects model on training IFC with grade, pretest IFC, condition (Alignment, No Alignment), and task (PN, NP) as fixed effects, and by-participant random intercepts to investigate whether alignment facilitated representational coherence. Results showed that pretest IFC predicted training IFC ($\beta = .62$, $SE = .06$, $p < .001$). Alignment significantly increased IFC compared with No Alignment (Alignment IFC $M = .91$, $SD = .24$; No Alignment IFC $M = .73$, $SD = .46$; $\beta = .38$, $SE = .11$, $p < .001$; Figure 8), indicating a more coherent representation between integers and fractions. There was no main effect of grade ($\beta = -.01$, $SE = .06$, $p = .999$), task ($\beta = .01$, $SE = .11$, $p = .951$), nor a task by condition interaction ($\beta = .18$, $SE = .21$, $p = .385$).

Similarly, we submitted training IFD to a mixed-effects linear model with grade, pretest IFD, condition (Alignment, No Alignment), and task (PN, NP) as fixed effects, by-participant random intercepts, by-participant random slope for pretest IFD, and by-item random intercepts. Consistent with IFC, pretest IFD predicted training IFD ($\beta = .23$, $SE = .05$, $p < .001$). Again, Alignment significantly decreased IFD (Alignment IFD $M = .05$, $SD = .14$; No Alignment IFD $M = .13$, $SD = .24$; $\beta = -.28$, $SE = .09$, $p < .01$). IFD was higher in the PN compared with NP task ($\beta = .16$, $SE = .03$, $p < .001$). The effect of condition significantly interacted with

[2] Alignment was dummy coded as −0.5, and No Alignment was dummy coded as 0.5.

Table 2

Mixed-Effects Regression Model Results on Percent Absolute Error for Training in Study 2

Predictors	*B*	*SE*	*df*	*t*	*p*
Integer PN					
(Intercept)	−0.04	0.07	20.98	−0.595	.558
Grade	−0.04	0.04	83.55	−0.965	.337
Pretest PAE	0.25	0.04	68.45	6.101	.0001***
Condition	−0.03	0.08	85.69	−0.372	.711
Integer NP					
(Intercept)	−0.06	0.07	16.54	−0.910	.376
Grade	−0.06	0.04	77.88	−1.814	.074
Pretest PAE	0.20	0.03	103.43	6.180	.0001***
Condition	0.37	0.07	78.11	5.242	.0001***
Fraction PN					
(Intercept)	−0.06	0.06	53.76	−1.011	.317
Grade	−0.05	0.04	50.58	−1.074	.288
Pretest PAE	0.23	0.04	109.51	6.487	.0001***
Condition	0.61	0.09	50.99	6.949	.0001***
Fraction NP					
(Intercept)	−0.07	0.06	29.85	−1.144	.262
Grade	−0.03	0.04	71.54	−.652	.516
Pretest PAE	0.23	0.03	108.34	7.570	.0001***
Condition	0.39	0.09	70.56	4.579	.0001***

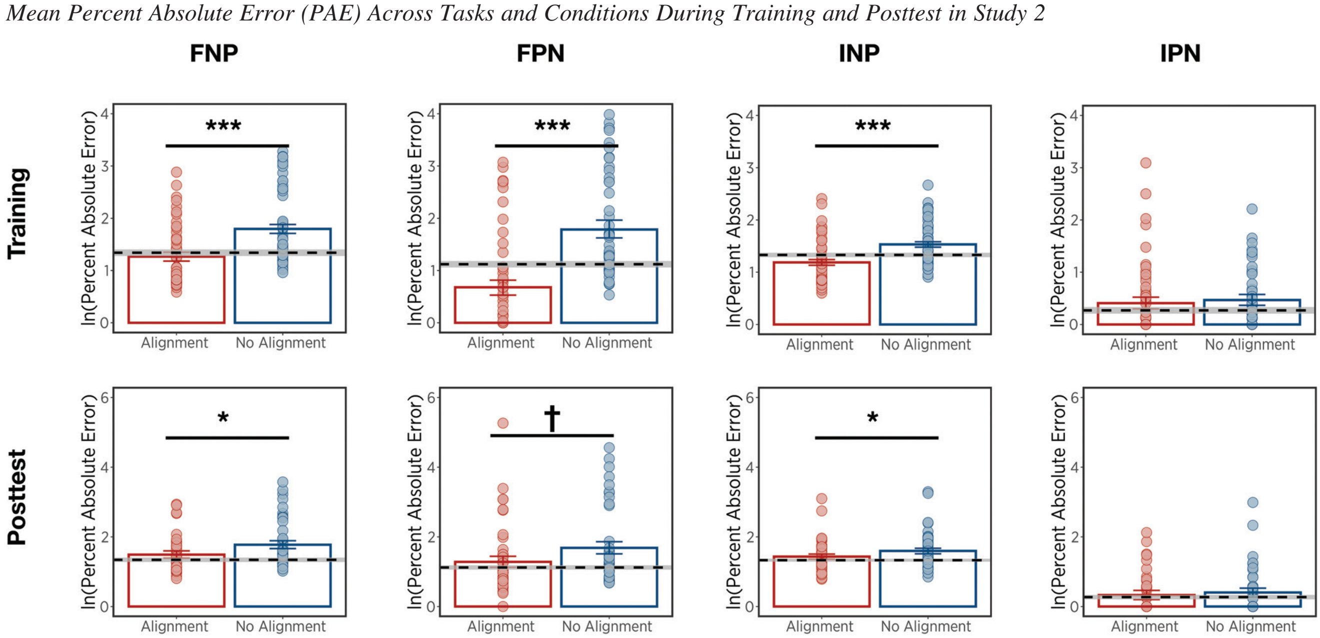
Note. B = slope; *SE* = standard error; *df* = degree of freedom; PN = position-to-number task; NP = number-to-position task; PAE = percent absolute error. The distribution of PAE was skewed, so we used log transformed and then standardized PAE as the dependent variable. Condition was coded as −.5 for Alignment and .5 for No Alignment condition.
*** $p < .001$.

task (β = −.17, *SE* = .06, $p < .01$), indicated by a larger difference between the Alignment and No Alignment group in the PN than NP (PN: Alignment IFD *M* = .06, *SD* = .18; No Alignment IFD *M* = .17, *SD* = .29; NP: Alignment IFD *M* = .04, *SD* = .10; No Alignment IFD *M* = .10, *SD* = .17;). There was no main effect of grade (β = −.07, *SE* = .05, *p* = .132).

We next compared children's representational coherence during training with that of college students. We found that children in

Figure 7

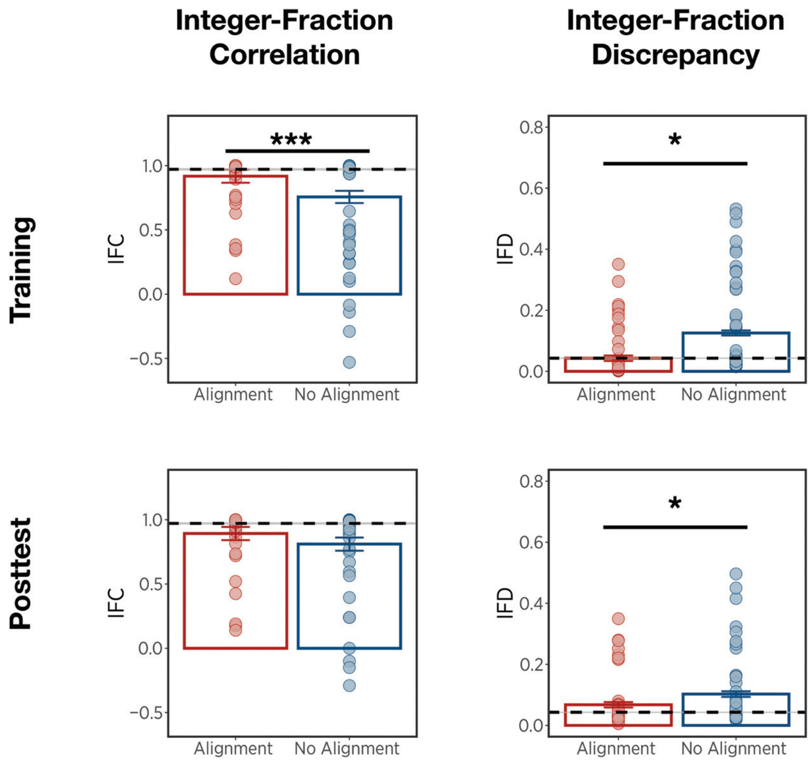
Mean Percent Absolute Error (PAE) Across Tasks and Conditions During Training and Posttest in Study 2



Note. Bars indicate children's predicted PAE from 1,000 simulations of the mixed linear modeling, adjusted for grade and pretest performance. Error bars indicate 95% confidence intervals. Dots indicate children's raw individual average PAE. Dotted lines indicate average PAE for the adults. Gray areas indicate standard errors. FNP = fraction number-to-position task; FPN = fraction position-to-number task; INP = integer number-to-position task; IPN = integer position-to-number task. See the online article for the color version of this figure.
† $p < .10$. * $p < .05$. *** $p < .001$.

Figure 8

Mean Integer-Fraction Correlation (IFC) and Integer-Fraction Discrepancy (IFD) Across Training and Posttest in Study 2



Note. Bars indicate children's predicted IFC/IFD from 1,000 simulations of the mixed linear modeling, adjusted for grade and pretest performance. Error bars indicate 95% confidence intervals. Dots indicate children's raw individual average IFC/IFD. Dotted lines indicate adults' average IFC/IFD. Gray areas indicate standard errors. IFC = Integer-Fraction Coherence; IFD = Integer-Fraction Discrepancy. * $p < .05$. *** $p < .001$. See the online article for the color version of this figure.

the Alignment group did not differ significantly from college students in terms of representational coherence (IFC: adults $M = .97$, $SD = .08$; $\beta = -.20$, $SE = .17$, $p = .229$; IFD: adults $M = .04$, $SD = .06$; $\beta = .03$, $SE = .13$, $p = .807$), whereas children in the No Alignment group produced significantly less coherent estimates than college students (IFC: $\beta = -.76$, $SE = .16$, $p < .001$; IFD: $\beta = .52$, $SE = .13$, $p < .001$), indicating that alignment boosted children's representational coherence to the same level as college students.

In summary, alignment between integers and fractions of proportional magnitudes led to more accurate estimates of fraction magnitude, more accurate estimates of integer magnitudes, and a more coherent representation of proportions overall. We next address whether this understanding transferred to a posttest in which the analogical source of integer number lines was unavailable.

Posttest

Seven participants (2 third graders, 3 fourth graders, and 2 fifth graders) who did not complete the posttest phase were excluded from further analyses. An additional one participant did not finish the Integer PN task, two participants did not finish the Integer NP task, one participant did not finish the Fraction NP task, and six participants did not finish the Fraction PN task on the posttest; thus, they were excluded from the following analyses on the corresponding task.

Parallel to our analyses for training, we implemented a series of linear mixed-effects models for each task separately with log-transformed PAE as the dependent variable and grade, pretest performance (measured by log-transformed PAE), and condition (Alignment, No Alignment) as fixed effects, a by-participant random slope for pretest performance, by-participant random intercepts, and by-item random intercepts.

Consistent with our findings for training, children who were more accurate when estimating integers and fractions on the pretest were also more accurate during the posttest (IPN: $\beta = .26$, $SE = .06$, $p < .001$; INP: $\beta = .27$, $SE = .04$, $p < .001$; FPN: $\beta = .32$, $SE = .04$, $p < .001$; FNP: $\beta = .31$, $SE = .04$, $p < .001$; Table 3). Children in the Alignment condition were significantly more accurate when estimating both fractions and integers than children in the No Alignment condition (IPN: Alignment PAE $M = 2.10\%$, $SD = .04$; No Alignment PAE $M = 2.72\%$, $SD = .04$; $\beta = .11$, $SE = .08$, $p = .172$; INP: Alignment PAE $M = 5.01\%$, $SD = .04$; No Alignment PAE $M = 6.12\%$, $SD = .06$; $\beta = .20$, $SE = .09$, $p < .05$; FPN:

Table 3
Mixed-Effects Regression Model Results on Percent Absolute Errors for Posttest in Study 2

Predictors	*B*	*SE*	*df*	*t*	*p*
Integer PN					
(Intercept)	−0.04	0.07	16.48	−0.500	.624
Grade	−0.04	0.04	68.59	−0.910	.366
Pretest PAE	0.26	0.06	63.95	4.546	.0001***
Condition	0.11	0.08	68.82	1.380	.172
Integer NP					
(Intercept)	−0.04	0.07	16.06	−0.509	.618
Grade	−0.01	0.05	58.50	−0.205	.838
Pretest PAE	0.27	0.04	92.52	6.985	.0001***
Condition	0.20	0.09	59.52	2.187	.033*
Fraction PN					
(Intercept)	−0.02	0.07	26.63	−0.235	.816
Grade	−0.10	0.05	30.26	−2.179	.037
Pretest PAE	0.32	0.04	81.92	7.166	.0001***
Condition	0.19	0.09	29.68	2.027	.052
Fraction NP					
(Intercept)	−0.06	0.06	18.53	−1.059	.303
Grade	−0.02	0.04	38.05	−0.546	.588
Pretest PAE	0.31	0.04	88.23	7.079	.0001***
Condition	0.21	0.07	38.75	2.819	.008*

Note. *B* = slope; *SE* = standard error; *df* = degree of freedom; PN = position-to-number task; NP = number-to-position task; PAE = percent absolute error. The distribution of PAE was skewed, so we log transformed and then standardized PAE as the dependent variable. Condition was coded as −.5 for Alignment and .5 for No Alignment condition.
* $p < .05$. *** $p < .001$.

Alignment PAE M = 18.01%, SD = .64; No Alignment PAE M = 22.99%, SD = .47; β = .19, SE = .09, p = .052; FNP: Alignment PAE M = 6.30%, SD = .07; No Alignment PAE M = 9.44%, SD = .11; β = .21, SE = .07, $p < .01$; Figure 7). This indicates that aligning fractions to integer number lines helps children better estimate fractional magnitudes, even when the analogical sources are not presented.

We then compared children's performance at posttest with that of college students to further examine the impact of training on tasks that tap a similar underlying construct, magnitude knowledge (see Figure 7). Children in the Alignment condition performed similarly to adults when estimating fractions (FNP: β = .17, SE = .13, p = .195; FPN: β = .16, SE = .15, p = .293) and integers (INP: β = .13, SE = .12, p = .273; IPN: β = .13, SE = .12, p = .282). In contrast, estimates of children in the No Alignment condition were less accurate than adults in all tasks (FNP: β = .47, SE = .13, $p < .001$; FPN: β = .48, SE = .15, $p < .01$; INP: β = .34, SE = .12, $p < .01$; IPN: β = .20, SE = .12, p = .084).

Does the effect of alignment on representational coherence persist at posttest? To test this, we conducted parallel analyses to the ones that we conducted on the training data. On posttest, there was no effect of alignment on IFC (Alignment IFC M = .89, SD = .26; No Alignment IFC M = .80, SD = .39; β = .11, SE = .12, p = .356). However, Alignment significantly decreased IFD (Alignment IFD M = .07, SD = .14; No Alignment IFD M = .10, SD = .20; β = −.17, SE = .09, $p < .05$). There was also an interaction effect between condition and task on IFD (β = −.20, SE = .08, $p < .01$), indicated by a larger difference between the Alignment and No Alignment group in the PN than NP task (PN: Alignment M = .07, SD = .15; No Alignment M = .12, SD = .25; NP: Alignment M = .07, SD = .12; No Alignment M = .09, SD = .15).

Condition may not have influenced IFC because children in the Alignment group were at ceiling on posttest. Children in the Alignment group did not differ significantly from college students in terms of representational coherence (IFC: β = −.31, SE = .18, p = .083; IFD: β = .20, SE = .13, p = .129); whereas children in the No Alignment group produced significantly less coherent estimates than college students (IFC: β = −.63, SE = .18, $p < .001$; IFD: β = .43, SE = .13, $p < .01$). Figure 8 reports covariate-adjusted IFC and IFD for children across groups compared with college students.

In summary, we found that children in the Alignment group were more accurate when estimating fractions on number lines and possessed more coherent representations of integers and fractions even at posttest when a direct alignment between integers and fractions was not visually presented. Moreover, children randomly assigned to the Alignment group produced estimates that were as accurate and as coherent as college students' estimates. These findings suggest that alignment between integers and fractions helps children grasp the underlying relational structure of fractions.

Discussion

Study 2 showed that alignment between integers and fractions on number lines yielded greater estimation accuracy for both fractions and integers, as well as more coherent and less discrepant representation of proportions. Posttraining tests indicated that the training effect persisted even when analogical sources desisted. Our results demonstrate that, even if children were not specifically instructed to make a comparison between the integer and fraction estimation problems, the vertical alignment between integer and fraction number lines afforded children the opportunity to spontaneously draw an analogy in which they could bootstrap their prior integer knowledge in an effort to make more accurate fraction estimates.

Our intervention showed substantial educational promise. In one short and unsupervised session, the simple intervention of aligning fractions and integers on number lines improved children's number-lines estimates to levels as good as, or even better than, college students. The overall effect size of our intervention session (d = .22–.56) were comparable to some past fraction intervention studies with multiple sessions (see Misquitta, 2011; Roesslein & Codding, 2019, for reviews). This overall effect size is probably an underestimate of the true training effect due to many older children reaching ceiling. For third graders, who had more improvement to make, effect sizes were substantially larger (d = .57–1.15).

General Discussion

The current studies investigated children's understanding of the implicit proportions denoted by ratios of integers and the explicit proportions denoted by fractions. The two studies demonstrated that children develop a more coherent representation of explicit and implicit proportions between third and fifth grade, that coherence predicts fraction proficiency (Study 1), and that comparing implicit and explicit proportions helps children to form a more coherent, accurate representation of explicit and implicit proportions (Study 2).

The Role of Integers in Fraction Learning

Our results strongly supported the idea that integer knowledge can be leveraged to positively influence fraction learning. In contrast to claims that integer knowledge and fraction knowledge develop separately (Gallistel & Gelman, 1992; Geary, 2006), our findings suggest that knowledge of integers does not necessarily interfere with the development of fraction understanding. Rather, the development of fractions and integers shared developmental continuities: children's integer and fraction estimates were positively correlated. This finding supports the integrated theories of numerical development (Siegler et al., 2011). Further, we examined *how* an integrated numerical understanding develops: (a) development of integer and fraction understanding is not independent, but rather the two interact and become coherent over development; and (b) children can bootstrap their understanding of fractions from their understanding of integers, if given the right cognitive support.

Mere practice with integers was not sufficient for children to improve their understanding of fractions. Consistent with structure-mapping theory (Gentner, 1983; Gentner & Markman, 1997), alignment between integer and fraction magnitudes on number lines highlights the common underlying relational structure between the given number and the total range, as well as the differences (e.g., each integer has a unique successor, but fractions do not have unique successors and can be placed between two adjacent integers on the number line). Therefore, alignment serves as cognitive support for children to boost their analogical transfer. Taken together, our correlational and experimental data are consistent with the idea that analogy serves as a learning mechanism that helps children extend their mental number line from familiar numbers to novel ones.

The Role of Analogy in Mathematical Development

This idea of structure mapping has also been proposed to explain how children learn to link symbolic numerals (e.g., 3) with their referents (e.g., 3 apples; Case et al., 1996; Sullivan & Barner, 2013, 2014). Namely, children learn the meaning of numbers based on mapping the relational structure of numbers to the structure of numerical quantities. In this view, knowing the meaning of "twenty" helps children learn the meaning of "forty" by structure mapping. For example, the ordinal relations of the two numbers (e.g., forty is twice as far into the count list as twenty) can be mapped to the cardinal relations of two numbers (forty denotes twice the numerosity of twenty). This mapping of structure provides a basis for understanding the cardinality of unfamiliar numbers whose ordinal meaning is known (a process known as "bootstrapping"; Carey, 2004; Gentner, 2010). Structure-based analogy, in comparison with item-based associative learning in which the meaning of each number needs to be linked with a mental representation of approximately this number of items (Lipton & Spelke, 2005), seems an effective way to accelerate numerical development because it enables children to "bootstrap" their limited experience with finite numbers to a potentially infinite range.

More broadly, knowledge of integer arithmetic can also lead to positive transfer to fraction arithmetic through analogies (Sidney, 2020; Sidney & Alibali, 2015, 2017; Sidney et al., 2019). To facilitate knowledge of fraction division, Sidney and Alibali (2015) provided fifth and sixth graders either a surface-level analog of another fraction operation or a structure-level analog of integer division. Students who were able to draw comparisons between division with integers and fractions gained more conceptual knowledge of fraction division than those who were able to draw comparisons between different fraction operations. Moreover, this positive transfer occurred even when the analogy between integer and fraction division was not explicitly instructed. In a follow-up study, Sidney and Alibali (2017) showed that solving integer division problems immediately before fraction division problems allowed students to analogically map between their integer and fraction knowledge, and improved learners' understanding of the underlying conceptual structure of division.

Furthermore, presenting information in a familiar, intuitive format (e.g., relations between integers) helps people solve challenging mathematical problems that involve proportion understanding. For example, adults were better able to solve Bayesian problems when the problems were presented in a frequency format that highlights relations of integers (e.g., 10 out of 1,000 women at age 40 have breast cancer) compared with a probability format with math symbols (e.g., the possibility of breast cancer is 1% for women at age 40; Gigerenzer, 1996; Gigerenzer & Hoffrage, 1995). Building a bridge between explicit proportions (e.g., 1%) and implicit proportions (e.g., 10 out of 1,000) can allow people to bootstrap their understanding of the less intuitive probability format from the more familiar frequency format. Sedlmeier and Gigerenzer (2001) demonstrated that those students who learned to reframe the probability format as the frequency format helped them to better learn and solve Bayesian problems, and the training effects lasted longer compared with the conventional way of teaching with only the probability format (Sedlmeier & Gigerenzer, 2001).

Limitations and Future Directions

Our findings have important educational implications. Number lines are a central conceptual structure that has been shown to be a useful tool in teaching children fractional magnitudes (Fazio et al., 2016; Hamdan & Gunderson, 2017; Moss & Case, 1999). With the cognitive support of visual alignment to support children as they attempt to draw analogies between less familiar fraction magnitudes and more familiar integers, children can rely on their already superior integer knowledge to learn fractions. Thus, classroom instruction can integrate a systematic comparison between the placement of integers and fractions on number lines to improve children's understanding of fractional magnitudes.

However, several issues were not addressed, including the breadth and duration of change in proportional reasoning. Ideally, we would test a much broader range of numbers on posttest and include these numbers in tests other than number-line estimation. Unfortunately, we were constrained by the fact that the training study needed to be short enough to administer in one session in a new online format at the start of the coronavirus disease 2019 (COVID-19) global pandemic. It would be useful to look at whether children can transfer their boosted fractional understanding to other fraction proficiency tasks beyond the number line estimation tasks, such as comparing and ordering fractions. Moreover, it would be useful to see how durable children's improved understanding of fractional magnitudes is. We anticipate that analogy can serve as a broad mechanism to expand the mental number line and improve fraction understanding underlying a variety of fraction proficiency tasks. We also anticipate that the training effect would last over time.

Future studies might also shed light on the moderators of fraction learning. For example, evidence from both longitudinal studies and computational modeling suggests that individual differences in executive functions predict children's analogical reasoning ability after controlling for age (Doumas et al., 2018; Morrison et al., 2011; Simms et al., 2018). Future studies can further explore how working memory updating and inhibitory control can moderate the effect of training.

Finally, although this study investigates only the effect of alignment, providing corrective feedback to children's estimates might further foster learning. Previous studies showed that interventions featuring feedback on children's estimates on fraction number lines facilitated a better understanding of fractional magnitudes (Fazio et al., 2016; Gunderson et al., 2019). Future studies may explore whether a training intervention combining both alignment and feedback yields an additive effect and even more improvements in fraction understandings.

Conclusions

Fractions are important to educational and professional success, but can be difficult for learners to grasp. However, novel mathematical concepts, like fractions, can be easier to understand if they build on children's foundational prior knowledge by leveraging analogies. The current study explored (a) the developmental trajectories of integer and fraction representations during a critical time when students in the United States learn fractions in their formal classroom lessons, and (b) whether alignment between more familiar integers and less familiar fractions represented on number lines would prompt children to draw analogies between these numerical concepts and improve fraction estimation performance. Our findings show that proportional representations of integers and fractions become more coherent over time, and alignment between integers and fractions facilitated more accurate fraction representations. In summary, our findings do not support the notion that integers must be seen as an obstacle to fraction learning. Rather, our findings suggest that preexisting, foundational integer knowledge can be a stepping stone toward an integration of integer and fraction knowledge when instruction draws on the power of analogies.

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